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Quadratic pseudosupersymmetry in two-level systems

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Abstract

Using the intertwining relation we construct a pseudosuperpartner for a (non-Hermitian) Dirac-like Hamiltonian describing a two-level system interacting in the rotating wave approximation with the electric component of an electromagnetic field. The two pseudosuperpartners and pseudosupersymmetry generators close a quadratic pseudosuperalgebra. A class of time-dependent electric fields for which the equation of motion for a two-level system placed in this field can be solved exactly is obtained. A new interesting phenomenon is observed. There exists a time-dependent detuning of the field frequency from the resonance value such that the probability to populate the excited level ceases to oscillate and becomes a monotonically growing function of time tending to $3/4$. It is shown that near this fixed excitation regime the probability exhibits two kinds of oscillations. The oscillations with small amplitude and frequency close to the Rabi frequency (fast oscillations) take place at the background of those with big amplitude and small frequency (slow oscillations). During the period of slow oscillations, the minimal value of the probability to populate the excited level may exceed $1/2$, suggesting for an ensemble of such two-level atoms the possibility of acquiring an inverse population and exhibit lasing properties.

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1. Introduction

Supersymmetry in physics was introduced in quantum field theory for unifying different interactions in a unique construct [1]. Supersymmetric formulation of quantum mechanics is due to the problem of spontaneous supersymmetry breaking [2]. The ideas of supersymmetry have been profitably applied to many nonrelativistic quantum mechanical problems since, and now there are no doubts that supersymmetric quantum mechanics (SUSY QM) exists in its own right (for recent developments see a special issue of 2004 *J. Phys. A: Math. Gen.* **34** (43)). It is worth noticing that most papers in this field deal with the Hermitian Hamiltonians.

A differential equation of Schrödinger-like type with a non-Hermitian Hamiltonian appears in many physical models. One can cite quantum systems coupled to the environment such as a hydrogen 'atom' in an interacting medium subject to a dissipative force [3] (see also [4]) or different decay or collision reactions (see, e.g., [5]; for more recent developments see [6]; in [7] the method of SUSY QM is involved). Physical requirements initiated a deep mathematical study of spectral problems with non-Hermitian Hamiltonians in the 1950s and 1960s. The most essential result was first obtained by Keldysh [8] who proved the completeness of the set of eigenfunctions and associated functions for a regular Sturm–Liouville problem with a non-Hermitian Hamiltonian. In the books by Naimark [9] and Marchenko [10] one can find good reviews of these studies.

A new impetus for studying different properties of non-Hermitian Hamiltonians is due to the discovery that the real character of the spectrum of a non-Hermitian Hamiltonian may be in particular related to so-called $\mathcal{PT}$-symmetry [11] and the suggestion to generalize quantum mechanics by accepting non-Hermitian Hamiltonians with a real spectrum to describe physical observables [12] (see also the review [13]). The necessary condition for such a generalization consists of the possibility of defining a Hilbert space with a positive definite metric which is intimately related to the property of a Hamiltonian to be diagonalizable (for recent discussions see, e.g., [14, 15]). This apparently may be assured in many cases since non-diagonalizable Hamiltonians may be transformed into diagonalizable ones by SUSY transformations [16]. The latter property permits us to suppose that the method of SUSY QM may become an essential ingredient of complex quantum mechanics. This conjecture is also supported by established properties of this method not only to offer the possibility of obtaining new exactly solvable complex potentials from known ones [17] but also to aid a deeper understanding of different properties of complex potentials [17, 18]. In particular, an explicit construction of a superalgebra involving non-Hermitian Hamiltonians, which may be useful in different contexts i.e. integrability, quantization, different quantum-field models etc, is shown to be possible [19] and even now has been extended to the notion of pseudosupersymmetry [20] and nonlinear pseudosupersymmetry [21].

The relation of the general two-level model described by a non-Hermitian Hamiltonian acting in the two-dimensional Hilbert space $\mathbb{C}^2$ with pseudosupersymmetry is discussed by Mostafazadeh [20]. In contrast to the approach of this author, we reduce the time-dependent Schrödinger equation for the two-level system, interacting in the rotating wave approximation with the electric component of an electromagnetic field, with a Hermitian Hamiltonian (see, e.g., [22]) to the one-dimensional stationary Dirac equation with an effective non-Hermitian Hamiltonian where time plays the role of the space variable. If we considered the spectral properties of the latter Hamiltonian we would define it in the Hilbert space $L^2(0, T) \otimes \mathbb{C}^2$. But as we shall see in our approach the spectral parameter in the Dirac equation is not related to spectral properties of the two-level system. Therefore we will not discuss any spectral features of this Hamiltonian and in particular its diagonalizability. Of course, the obtained Dirac equation is completely equivalent to the initial Schrödinger equation, and if one studied it by usual means one would not get any new information about the two-level system. From this point of view, the method of SUSY QM we are using proves its extreme efficiency once again.

To find a pseudosuperpartner for the given Dirac-like Hamiltonian, we are using the technique of intertwining operators developed in [23] for the one-dimensional stationary Dirac equation. We have to note that the application of results of this paper to our particular problem is not straightforward since transformation operators of the general form do not preserve the very peculiar form of the effective Dirac Hamiltonian corresponding to the two-level system. So, below we show how one can choose the necessary ones from the wide variety

of possible transformations. In our approach in contrast to [20] the two pseudosuperpartners and pseudosupersymmetry generators constructed with the help of first-order intertwiners close a quadratic pseudosuperalgebra. As usually happens for the method of intertwining operators [24], if one of the two Hamiltonians is exactly solvable the same property occurs for the other. In this way starting from the simplest case corresponding to the famous Rabi oscillations, we have found new electric fields having time-dependent frequencies for which the equation of motion of the two-level system has exact solutions. While analysing solutions of the Schrödinger equation, we have found a new interesting physical phenomenon. We show that there exists a time-dependent detuning of the field frequency from the resonance value such that the probability to populate the excited level ceases to oscillate and becomes a monotonically growing function of time tending to $3/4$. Of course this is a strictly fixed excitation regime similar to resonance. We also study how the above probability behaves under small deviations from this specific regime. We have found that when the parameters of the model are close enough to the specific values, the probability exhibits two kinds of oscillations. The oscillations with small amplitude and frequency close to the Rabi frequency (fast oscillations) take place at the background of those with big amplitude and small frequency (slow oscillations). During the period of slow oscillations, which grows when the parameters of the model approach the above specific values, the minimal value of the probability to populate the excited level may exceed $1/2$, suggesting for an ensemble of such two-level atoms the possibility of acquiring an inverse population and exhibit lasing properties.

We have to note that some of the results we give below are known from a previous paper [25]. These authors also use a similar intertwining technique but they do not relate it to pseudosupersymmetry and do not give any analysis of solutions this method can provide. Moreover, we give a deeper analysis of restrictions imposed on transformation operators by the features of the two-level system. In particular, we show that both the new Hamiltonian and solutions of the new Dirac equation can be expressed in terms of a real-valued function which is a solution of a second-order differential equation with real coefficients. Since such equations always have real solutions our analysis opens up the direct possibility of realizing chains of transformations preserving the form of the Dirac-like Hamiltonian imposed by the features of the two-level system.

2. Preliminary

The two-level model in the rotating wave approximation with a possibly time-dependent detuning is described by the following system of equations (see, e.g., [22]):

$$\mathrm{i}\dot{A}_1 - fA_1 = \xi A_2 \qquad \mathrm{i}\dot{A}_2 + fA_2 = \xi A_1 \tag{1}$$

where $\xi = \frac{1}{2\hbar}E_0 d_{12}$, d_{12} is the matrix element of the dipole interaction operator, E_0 is the amplitude of the electric component of an external electromagnetic field; $f = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\delta t)$, $\delta(t) = \omega_{12} - \omega(t)$, $\omega_{12} = \frac{1}{\hbar}(\varepsilon_1 - \varepsilon_2)$, ε_1 and ε_2 are energy levels of the free atom and $\omega(t)$ is the field frequency; the dot over the symbol means the derivative with respect to time. While normalized properly the functions $|A_1(t)|^2$ and $|A_2(t)|^2$ give occupation probabilities for the ground and excited states respectively. If ω does not depend on time (hence $f = \frac{1}{2}\delta = \mathrm{const}$), the solutions of system (1) are well known. For instance, with the initial condition $A_2 = 0$ and $A_1 = 1$ at $t = 0$ we get the well-known formula [22] for the excited state occupation probability if initially the system is in the ground state

$$P(t) = |A_2(t)|^2 = \frac{\xi^2}{2\Omega^2}\left[1 - \cos(2\Omega t)\right] \qquad \Omega^2 = f^2 + \xi^2 \tag{2}$$

with 2ξ known as the Rabi frequency. Probability (2) is an oscillating function of time (so-called Rabi oscillations). At the resonance $\left(f = \frac{1}{2}\delta = 0\right)$ it oscillates with the Rabi frequency. Therefore the value $\delta(t)$ characterizes the detuning of $\omega(t)$ from its resonance value equal to ω_{12}. In section 5 using the formalism developed in section 4 we shall get time-dependent functions $f = f(t)$ (and hence $\delta(t)$) for which system (1) permits exact solutions. As we show below (section 5), there are time-dependent corrections to the detuning that we will consider; although they may change crucially the time-dependent behaviour of the solutions of system (1), they essentially keep the oscillating character of the probability to populate the excited level with the frequency close to 2Ω. Yet, the absence of the Rabi oscillations may be considered as oscillations with the same frequency but with the zero amplitude since they may be obtained as the corresponding limiting case of oscillations with a nonzero amplitude. So, in our approach the rotating wave approximation is as good as it is in the classical case of an electric field of constant frequency.

Let us rewrite system (1) in the matrix form

$$h_0\Psi = E\Psi \qquad h_0 = \gamma\partial_t + V_0 \tag{3}$$

where

$$V_0 = \mathrm{i}f_0\sigma_y \tag{4}$$

$\gamma = \mathrm{i}\sigma_x$, $E = \xi$, $\Psi = (A_1, A_2)^{\mathrm{T}}$ (the superscript 'T' denotes the transposition) and we replaced f (which we will call the 'potential') in (1) by f_0; $\sigma_{x,y,z}$ denote the standard Pauli matrices. Equation (3) is the one-dimensional stationary Dirac equation with the non-Hermitian Hamiltonian h_0 defined by potential (4) where t plays the role of the space variable. By construction the parameters f_0 and E are real. For a fixed value of the dipole momentum of the irradiated system, the parameter $E = \xi$ is defined by the amplitude of the electric field and, hence, is not related to spectral properties of the system. A useful comment is that since the Hamiltonian of system (1) is Hermitian, $H_{\mathrm{sch}} = \left(\begin{smallmatrix} f & \xi \\ \xi & -f \end{smallmatrix}\right)$, the evolution of the two-level system is unitary even for a time-dependent function $f = f(t)$. This means that the $\mathbb{C}^2$ inner product, $|A_1(t)|^2 + |A_2(t)|^2$, for the Dirac equation (3) is t-independent.

3. SUSY algebra with non-Hermitian Hamiltonians

Let us have a non-Hermitian Hamiltonian h_0. We will not consider it as a Hamiltonian acting in a Hilbert space, but to construct a SUSY algebra we need adjoint operators which we will introduce in a formal way. Denote by h_0^+ the operator formally adjoint to h_0. As usual the adjoint operation consists of taking the complex conjugation and transposition; the operator of the first derivative is skew-Hermitian and $(AB)^+ = B^+A^+$.

Let h_1 be a 'transformed Hamiltonian' which should be found together with the transformation operator L by solving the intertwining relation $Lh_0 = h_1L$ and h_1^+ be its adjoint. The later participates in the adjoint intertwining relation $h_0^+L^+ = L^+h_1^+$. It means that the operator L^+ transforms eigenfunctions of h_1^+ into eigenfunctions of h_0^+.

Let us suppose that there exists an operator J such that $h_{0,1}^+ = Jh_{0,1}J$ and $J^2 = \pm 1$, $J^+ = \pm J$ (in general both signs may be accepted). Then from the adjoint intertwining relation, it follows that $JL^+Jh_1 = h_0JL^+J$ meaning that the operator JL^+J realizes the backward transformation from h_1 to h_0 and the operator JLJ transforms from h_0^+ to h_1^+. From here we infer that the superposition JL^+JL transforms solutions of equation (3) into solutions of the same equation. This means that this is a symmetry operator for this equation. In the simplest case when L is a differential operator that we would like to consider that this symmetry operator may be a function of h_0, so we will suppose that $JL^+JL = F_1(h_0)$. For the same reason,

the superposition LJL^+J may be a function of h_1 leading to $LJL^+J = F_2(h_1)$. Moreover, we will also suppose that $F_2(x) = F_1(x) \equiv F(x)$ is an analytic function. These properties generalize the known factorization (polynomial factorization if $F(x)$ is a polynomial, see e.g. [23, 24]) properties taking place for the Hermitian case.

It follows from (3) and (4) that in our case $J = \sigma_x$. Keeping in mind the properties of the operators L and J let us introduce the following matrix operators:

$$H = \begin{pmatrix} h_0 & 0 \\ 0 & h_1 \end{pmatrix} \qquad Q_1 = \begin{pmatrix} 0 & 0 \\ L & 0 \end{pmatrix} \qquad Q_2 = \begin{pmatrix} 0 & JL^+J \\ 0 & 0 \end{pmatrix}. \tag{5}$$

It follows from the intertwining relations that the operators Q_1, and Q_2 commute with H and they apparently are nilpotent. The above factorization properties are equivalent to the following anticommutation relation: $Q_1Q_2 + Q_2Q_1 = F(H)$.

Now if we identify our J operator with $\eta_- = \eta_+^{-1}$ introduced in [20], $J = \eta_- = \eta_+^{-1}$, our L operator with D and JL^+J with $D^\sharp$, we conclude that the operator Q_2 becomes pseudoadjoint to Q_1, the operators H, Q_1 and Q_2 close a nonlinear superalgebra and one can associate a nonlinear pseudosupersymmetry with a quantum system described by the Hamiltonian H. In the following section, we shall show that a quadratic pseudosupersymmetry may be associated with the two-level system.

4. Intertwining operators for two-level Hamiltonians

To be able to associate a pseudosupersymmetry with the Hamiltonian given in (3) and (4), we have to find an intertwining operator and a partner Hamiltonian h_1. According to [23] the intertwining operator L for a matrix equation such as (3) is defined with the help of a matrix-valued function $\mathcal{U} = \mathcal{U}(t)$ satisfying the equation

$$h_0\mathcal{U} = \mathcal{U}\Lambda \qquad \Lambda = \mathrm{diag}(\lambda_1, \lambda_2) \tag{6}$$

called the 'transformation function', as follows:

$$L = \partial_t - W \qquad W = \dot{\mathcal{U}}\mathcal{U}^{-1}. \tag{7}$$

Here λ_1 and λ_2 are arbitrary constants. The operator L transforms a solution Ψ of equation (3) into a solution Φ of the same equation where the matrix V_0 is replaced by

$$V_1 = V_0 + \Delta V \qquad \Delta V = \gamma W - W\gamma. \tag{8}$$

Here and in the following the subscript 0 marks quantities before the transformation and 1 marks these after the transformation. It is not difficult to see that to preserve form (4) of the potential so that $V_1 = \mathrm{i}f_1\sigma_y$, it is sufficient to take the transformation function of the form

$$\mathcal{U} = \begin{pmatrix} u_{11} & u_{11} \\ u_{21} & -u_{21} \end{pmatrix}. \tag{9}$$

In this case the column-vector $U_1 = (u_{11}, u_{21})^{\mathrm{T}}$ is a solution to the initial equation (3) corresponding to the eigenvalue λ and the column-vector $U_2 = (u_{11}, -u_{21})^{\mathrm{T}}$ is a solution to the same equation with the eigenvalue $-\lambda$ (note that this symmetry is built into the system (3)!) so that Λ in (6) has the form $\Lambda = \mathrm{diag}(\lambda, -\lambda)$. After some simple algebra one finds from (8) that $f_1 = f_0 + \Delta f$ where

$$\Delta f = \lambda\left(\frac{u_{11}}{u_{21}} - \frac{u_{21}}{u_{11}}\right) - 2f_0. \tag{10}$$

In general, solutions $U_{1,2}$ of equation (3) from which the matrix $\mathcal{U}$ is composed, $\mathcal{U} = (U_1, U_2)$, are complex, leading to a complex-valued potential difference Δf. For

physical reasons, we require real potentials. A necessary condition for Δf to be real is that the eigenvalue λ be purely imaginary. Indeed, it is easy to show that λ cannot be real. According to (10) Δf is defined by the expression $\frac{u_{11}}{u_{21}} - \frac{u_{21}}{u_{11}}$. Putting $\frac{u_{11}}{u_{21}} = \varrho \exp(\mathrm{i}\varphi)$ one finds

$$\frac{u_{11}}{u_{21}} - \frac{u_{21}}{u_{11}} = \left(\varrho - \frac{1}{\varrho}\right)\cos\varphi + \mathrm{i}\left(\varrho + \frac{1}{\varrho}\right)\sin\varphi \tag{11}$$

and our claim follows from the fact that $\varrho + \frac{1}{\varrho}$ is never equal to zero. Finally one can prove that λ^2 is real (cf [25]).

Now when the imaginary character of λ is established we see from (10) that the left-hand side of (11) must be purely imaginary, which is possible only if $\varrho = 1$, meaning that u_{11} and u_{21} have the same absolute value. Therefore one can put $u_{11} = \rho \exp(\mathrm{i}\varphi_1)$ and $u_{21} = \rho \exp(\mathrm{i}\varphi_2)$. Using the fact that $U_1 = (u_{11}, u_{21})^{\mathrm{T}}$ satisfies equation (3) with $E = \lambda$ and setting $\lambda = \mathrm{i}R$, where R is real, one gets from (3) a system of equations for ρ, φ_1 and φ_2. Of these equations we need only

$$\dot\varphi_2 - \dot\varphi_1 - 2f_0 + 2R\sin(\varphi_2 - \varphi_1) = 0. \tag{12}$$

If $R = 0$, equation (12) can readily be integrated. Suppose $R \neq 0$, the change of the dependent variable in equation (12), $\varphi_2 - \varphi_1 = 2\arctan q$, yields for q the Riccati equation

$$\dot q + 2Rq - f_0(1 + q^2) = 0. \tag{13}$$

If $f_0 = 0$ the equation for q is readily integrated: $q = \exp(-2Rt)$. Considering $f_0 \neq 0$ one can linearize (13) by putting $q = -\dot u/(uf_0)$, so u is a solution to the second-order equation

$$\ddot u + (2R - \dot f_0/f_0)\dot u + f_0^2 u = 0. \tag{14}$$

Introducing the new variable ψ by putting $u = \exp(-Rt)\sqrt{f_0}\psi$, one eliminates the first derivative term from (14) thus obtaining

$$\ddot\psi + \left[f_0^2 + \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\ln f_0 - \left(\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\ln f_0 - R\right)^2\right]\psi = 0. \tag{15}$$

This equation has two linearly independent real solutions and, hence, ψ is defined up to one real constant. Once ψ is fixed one calculates q:

$$q = \frac{R}{f_0} - \frac{\dot f_0}{2f_0^2} - \frac{\dot\psi}{f_0\psi} \tag{16}$$

and the potential difference $\Delta f = 2R\sin(\varphi_2 - \varphi_1) - 2f_0$:

$$\Delta f = \frac{4Rq}{1 + q^2} - 2f_0. \tag{17}$$

Solution Φ of the equation $h_1\Phi = E\Phi$ with $h_1 = \gamma\partial_t + V_1$, $V_1 = V_0 + \Delta V$, $\Delta V = \mathrm{i}\Delta f\sigma_y$ can be found by applying the transformation operator (7) to the solution Ψ of equation (3), $\Phi = L\Psi$. It is easy to see that the matrix W is diagonal

$$W = \mathrm{diag}(w_1, w_2) \qquad w_1 = -\mathrm{i}f_0 + Ru_{21}/u_{11} \qquad w_2 = w_1^*. \tag{18}$$

and the ratio of the components of the spinor U_1 defining w_1 in (18) is also expressible in terms of the function q:

$$\frac{u_{21}}{u_{11}} = \frac{(1 + \mathrm{i}q)^2}{1 + q^2}. \tag{19}$$

Finally, skipping calculational details but noting that just in the same way as was done in [23], one can find the following factorizations:

$$JL^+JL = h_0^2 - \lambda^2, \qquad LJL^+J = h_1^2 - \lambda^2 \tag{20}$$

with $J = \sigma_1$. This means that the function F from section 3 is $F(x) = x^2 - \lambda^2$, the operators H, Q_1 and Q_2 close the quadratic superalgebra, and the quadratic pseudosupersymmetry underlies the two-level system interacting with the electric component of an electromagnetic field.

5. Application: SUSY transformations of the Rabi oscillations

In this section we illustrate a new physical phenomenon we observed while analysing solutions of system (1) obtained using the above-developed technique.

We start with $\delta_0 = 2f_0 = \omega_{12} - \omega_0 = \text{constant}$ (this corresponds to the Rabi oscillations (2)) to get a time-dependent 'potential' $f_1(t) = f_0 + \Delta f(t) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}[\delta_1(t)t]$. Once $f_1(t)$ is found we calculate the detuning $\delta_1(t) = \omega_{12} - \omega_1(t)$ by integrating the previous equation

$$\delta_1(t) = \frac{2}{t}\int_0^t f_1(t)\,\mathrm{d}t. \tag{21}$$

We have found that relatively small but time-dependent perturbations of the field frequency $\omega_1(t)$ from its resonance value equal to ω_{12} may influence essentially the time behaviour of the probability $P_1(t)$ to populate the excited state level with respect to the constant frequency case.

If $f_0 = \text{const}$, equation (15) for ψ reduces to

$$\ddot{\psi} + \varpi^2\psi = 0 \qquad \varpi^2 = f_0^2 - R^2 = \text{const}. \tag{22}$$

Solutions of this equation have different properties depending on whether the value ϖ^2 is positive, negative or zero. We have found that the oscillating behaviour of the probability $P_1(t)$ disappears when $\varpi = 0$. In this case the general solution to equation (22) is a linear function of time $\psi = At + B$ which according to (16) gives the following time dependence of the function q: $q(t) = 1 - A/(Atf_0 + Bf_0)$. Once $q(t)$ is found one calculates the 'potential difference' with the help of formula (10) and finally the new 'potential' $f = f_1(t)$:

$$f_1(t) = f_0 - \frac{2A^2 f_0}{2A^2 f_0^2 t^2 - 2Af_0(A - 2Bf_0)t + A^2 - 2ABf_0 + 2B^2 f_0^2}. \tag{23}$$

Another restriction leading to the desired result is $A = 2Bf_0$ which reduces the previous equation to a simpler form

$$f_1(t) = f_0 - \frac{4f_0}{1 + 4f_0^2 t^2}. \tag{24}$$

Since solutions $A_{10}(t)$ and $A_{20}(t)$ of system (1) for $f = f_0 = \text{const}$ are known, one can find solutions $A_{11}(t)$ and $A_{21}(t)$ of the same system with $f = f_1(t)$ by applying the transformation operator L defined by formulae (7), (18) and (19) to the previous solution. In this way, imposing the initial condition $A_{11}(0) = 1$ and $A_{21}(0) = 0$ one finds the probability $P_1(t)$ to populate the excited level at the time moment t if at $t = 0$ only the ground state level is populated

$$P_1(t) = |A_{21}(t)|^2 = \frac{\xi^2}{\Omega_0^6\left(1 + 4f_0^2 t^2\right)}\Big[16f_0^4\Omega_0^2 t^2\cos^2\Omega_0 t + 4f_0^2\Omega_0 t\left(\xi^2 - 3f_0^2\right)\sin 2\Omega_0 t + \left(4f_0^2\Omega_0^4 t^2 + \left(\xi^2 - 3f_0^2\right)^2\right)\sin^2\Omega_0 t\Big]. \tag{25}$$

Here $\Omega_0 = \sqrt{f_0^2 + \xi^2}$ and $2\Omega_0$ is the frequency of oscillations of the probability $P_0(t)$ (2) at $f = f_0$. It is clearly seen that $P_1(t)$ is an oscillating function provided $\xi^2 \neq 3f_0^2$. For $\xi^2 = 3f_0^2$ ($\Omega_0 = 2f_0$) the probability becomes equal to

$$P_1(t) = \frac{3f_0^2 t^2}{1 + 4f_0^2 t^2} \tag{26}$$

which is a function monotonically growing from zero at the initial time moment up to the value $3/4$ at $t \to \infty$. We have to note that for a fixed ξ the parameter f_0 is fixed also, $f_0 = \xi/\sqrt{3}$, which by means of formulae (24) and (21) fixes the frequency of the electric field in a unique way. So, for the given dipole momentum this excitation regime is fixed by the amplitude of the electric field. Let us analyse now what is happening with the probability $P_1(t)$ when the parameters of the model are close to this exceptional point.

Suppose now $\varpi^2 > 0$ and we will consider it to be close to zero. In this case the general solution to equation (22) may be written as $\psi = \frac{A}{\varpi}\sin(\varpi t + a + b)$. The function q as given in (16) does not depend on the value of the coefficient $\frac{A}{\varpi}$ but we need this coefficient to realize the limit $\varpi \to 0$ thus recovering the previously obtained solution. Choosing b such that $\sin 2b = \varpi/f_0$ and $\cos 2b = R/f_0$ but keeping a arbitrary one gets

$$\frac{\dot{\psi}}{\psi} = -\varpi\frac{\varpi - f_0\sin(2\varpi t + 2a)}{R - f_0\cos(2\varpi t + 2a)}. \tag{27}$$

This leads to the following expression for q:

$$q = \frac{R\cos(2\varpi t + 2a) + \varpi\sin(2\varpi t + 2a) - f_0}{f_0\cos(2\varpi t + 2a) - R} \tag{28}$$

and finally to the 'potential difference' of the form

$$\Delta f(t) = \frac{2\varpi^2}{R\cos(2\varpi t + 2a) - f_0}. \tag{29}$$

This formula has been previously derived by Bagrov *et al* by other means [25]. Putting $a = \operatorname{arctg}\frac{\varpi}{2f_0} - \frac{1}{2}\operatorname{arctg}\frac{\varpi}{R}$ one recovers for $f_1(t) = f_0 + \Delta f(t)$ the previous result (24) as the limit $\varpi \to 0$. This means that for ϖ close to zero the probability $P_1(t)$ corresponding to the potential difference (29) should be close to the previous value (26). The analytic expression for $P_1(t)$ is rather complicated and we will restrict ourselves to graphical illustrations.

Let us fix the Rabi frequency 2ξ. The function $\Delta f(t)$ (29) contains three parameters ϖ, f_0 and a. The parameter f_0 defines the value $2\Omega_0 = 2\sqrt{f_0^2 + \xi^2}$, which is the frequency of oscillations of the function $P_0(t)$ given by (2) to which $P_1(t)$ is reduced when the time-dependent correction $\Delta f(t)$ is absent. As was already mentioned when $f_0 = 0$ (resonance case) the function $P_0(t)$ oscillates with the Rabi frequency 2ξ. The parameter ϖ defines the frequency of the time-dependent correction $\Delta f(t)$ (29) for $f_1 = f_0 + \Delta f$, and the parameter a is responsible for the initial value of $f_1(t)$. The probability $P_1(t)$ is a periodical function if Ω_0 is commensurable with ϖ. In this case it exhibits two kinds of oscillations, namely, fast oscillations with frequency $2\Omega_0$, which is close to the Rabi frequency when f_0 is close to zero, taking place at the background of slow oscillations with frequency 2ϖ.

For our numerical illustrations we choose $f_0 = 1$. If in standard units this is $1 \times 10^{11}\ \mathrm{c}^{-1}$ this corresponds to 10^{-11} c as the unit of time in our figures.

Figure 1(a) shows the probability $P_1(t)$ for $\Omega_0 = 2$, $a = 0.015$ and $\varpi = 1/4$ (solid line) and $\varpi = 1/6$ (dotted line).

Figure 1(b) illustrates the time behaviour of the detuning $\delta_1(t)$ calculated according to (21) for $a = 0.015$, $\varpi = 1/4$ and $\varpi = 1/6$ (solid and dotted lines respectively) together with its limiting value corresponding to $\varpi = 10^{-3}$ and $a = 10^{-6}$ (dashed line). It is clearly seen from figure 1(a) that the period of slow oscillations grows when ϖ decreases and fast oscillations go around the limiting value 0.75 with the amplitude increasing with ϖ decreasing. Moreover, figure 1(b) says that the oscillating behaviour of $P_1(t)$ is transformed into monotonically growing one when for $\varpi = 0$ the detuning becomes a monotonic function of time (dotted line in figure 1(b)). If it acquires some oscillating perturbations the probability also starts to oscillate.

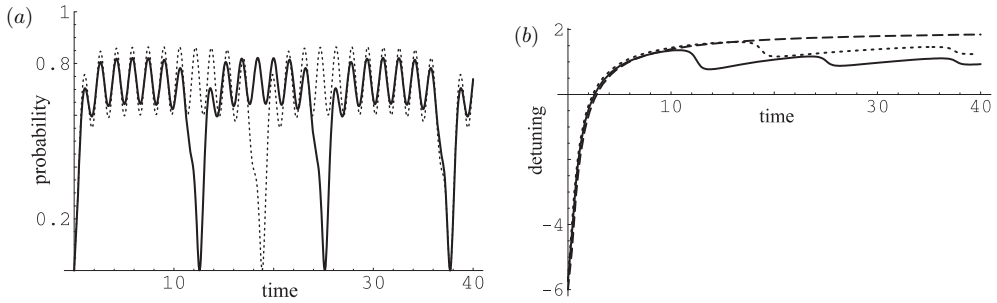

Figure 1. (a) Probability $P_1(t)$ at different values of ϖ. (b) Detuning $\delta_1(t)$ at different values of ϖ.

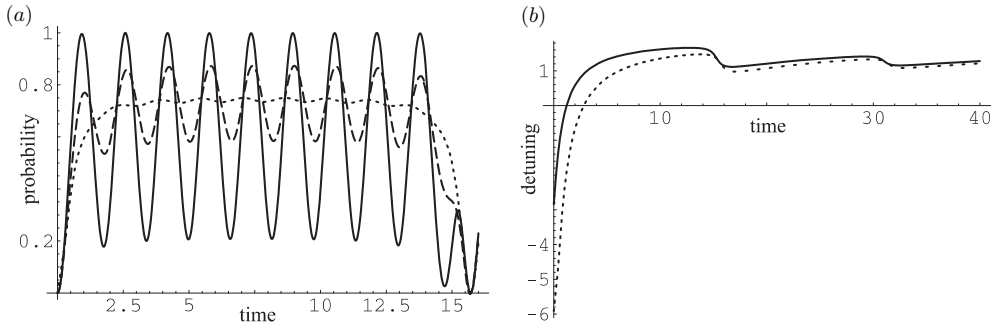

Figure 2. (a) Probability $P_1(t)$ at different values of a. (b) Detuning $\delta_1(t)$ at different values of a.

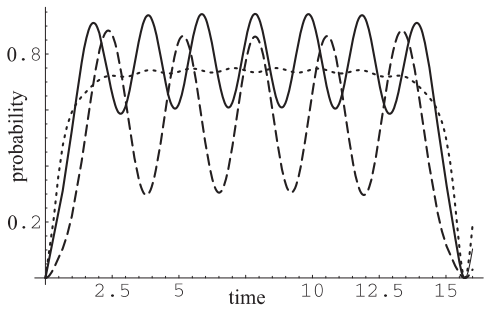

Figure 3. Probability $P_1(t)$ at different values of Ω_0.

Figures 2 and 3 show the dependence of the same quantities on the parameter a which is responsible for the phase shift in formula (29) at the fixed value $\varpi = 1/5$. Dotted, dashed and solid lines (figure 2(a)) correspond to $a = 0$, $a = 0.02$ and $a = 0.08$ respectively. Figure 2(b) shows the time dependence of the detuning $\delta(t)$ for $a = 0$ (dotted line) and $a = 0.08$ (solid line). From figure 2(b) we can conclude that the parameter a defines mainly the maximum of the absolute value of the detuning which it takes at $t = 0$. Figure 2(a) says that the amplitude of fast oscillations grows together with a.

Figure 3 shows the dependence of $P_1(t)$ from the frequency of fast oscillations Ω_0 at $a = 0$ and $\varpi = 0.2$. Dotted, solid and dashed lines correspond to $\Omega_0 = 2$, $\Omega_0 = 1.6$ and $\Omega_0 = 1.2$ respectively. The more it differs from the critical value equal to 2 corresponding

to $\xi^2 = 3f_0^2$, when the oscillations in formula (25) disappear, the bigger the amplitude of the fast oscillations becomes.

6. Conclusion

Using the technique of intertwining operators for a Dirac-like system developed in [23] we have found time-dependent electric fields for which the equation of motion for a two-level system placed in this field obtained after the rotating wave approximation can be solved exactly. Pseudosupersymmetry generators constructed with the help of intertwining operators together with the super-Hamiltonian close a quadratic deformation of the superalgebra constructed in [20]. We conclude, hence, that two-level systems in external electromagnetic fields may have hidden quadratic pseudosupersymmetry which is responsible for the new phenomenon consisting of disappearance of the Rabi oscillations.

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